J. Math. Anal. Appl. 270 (2002) 230–246

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.academicpress.com

Singular solutions of a nonlinear equation in bounded domains of R^2

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Received 2 May 2001

Submitted by G. Hagedorn

Abstract

We consider the nonlinear problem

$$(\mathrm{P})\begin{cases}\Delta u(x)+f(x,u(x))=0, & x\in D\setminus\{0\},\\ u(x)>0, & x\in D\setminus\{0\},\\ u(x)\sim \operatorname{Log}1/|x| \text{ near } x=0,\\ u(x)=0, & x\in\partial D,\end{cases}$$

where D is a bounded regular Jordan domain in R^2 containing 0 and f is a measurable function on $D\times(0,\infty)$. When the function $x\to f(x,G(x,0))/G(x,0)$ is in a certain class K, we show the existence of infinitely many solutions of (P).

$G(x,y)$ is the Green's function of the Laplacian in D. © 2002 Elsevier Science (USA). All rights reserved.

Résumé

On considére le problème non-lineaire suivant :

$$(\mathrm{P})\begin{cases}\Delta u(x)+f(x,u(x))=0, & x\in D\setminus\{0\},\\ u(x)>0, & x\in D\setminus\{0\},\\ u(x)\sim \operatorname{Log}1/|x| \text{ au voisinage de } x=0,\\ u(x)=0, & x\in\partial D,\end{cases}$$

où D est un domaine de Jordan borné et régulier contenant 0 et f est une fonction mesurable sur $D\times(0,\infty)$. On montre que si la fonction $x\to f(x,G(x,0))/G(x,0)$ est dans une certaine classe K, alors le problème (P) admet une infinité de solutions.

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PII: S0022-247X(02)00069-0

$G(x, y)$ est la fonction de Green du Laplacian sur D.

1. Introduction

In [5], Zhang and Zhao studied the following problem

$$(*)\begin{cases} \Delta u(x) + V(x)u^p(x) = 0, \ x \in D \setminus \{0\}, \\ u(x) > 0, \ x \in D \setminus \{0\}, \\ u(x) \sim 1/|x|^{n-2} \text{ near } x = 0, \\ u(x) = 0, \ x \in \partial D, \end{cases}$$

where $D \subset R^n$ $(n \geqslant 3)$ is a bounded Lipchitz domain containing 0, $p > 1$ and V is a measurable function such that $V(x)/|x|^{(n-2)(p-1)}$ is in the Kato class K_n.

Then, they showed the existence of infinitely many solutions of $(*)$.

In this paper, we prove that the existence of infinitely many singular solutions is valid for the following nonlinear problem

$$(\mathrm{P})\begin{cases} \Delta u(x) + f(x, u(x)) = 0, \ x \in D \setminus \{0\}, \\ u(x) > 0, \ x \in D \setminus \{0\}, \\ u(x) \sim \operatorname{Log} 1/|x| \text{ near } x = 0, \\ u(x) = 0, \ x \in \partial D, \end{cases}$$

where D is a bounded regular Jordan domain in R^2 containing 0, and f is a measurable function on $D \times (0, \infty)$.

The notion $u(x) \sim \operatorname{Log} 1/|x|$ near $x = 0$ means that for some $C > 0$,

$$\lim_{|x| \to 0} \frac{u(x)}{\operatorname{Log} 1/|x|} = C.$$

Solutions of this problem are understood as distributional solutions in D.

Let us introduce the conditions on the function f. It turns out that these conditions are relating to the next class K which properly contains the classical Kato class K_2 (for the properties of K_2, see Section 2, [1] or [2]).

Definition 1.1. A Borel measurable function φ in D belongs to the Kato class K if φ satisfies the following condition

$$\lim_{\alpha \to 0}\left(\sup_{x \in D} \int_{(|x-y| \leqslant \alpha) \cap D} \frac{\rho(y)}{\rho(x)} \operatorname{Log}\left(1 + \frac{\rho(x)\rho(y)}{|x-y|^2}\right) \big|\varphi(y)\big|\, dy\right) = 0, \tag{1.1}$$

where $\rho(x)$ is the distance from x to ∂D.

The following hypotheses on f are adopted.

(H_1) f is a measurable function on $D \times (0,\infty)$, continuous with respect to the second variable and satisfies

$$\left|f(x,t)\right| \leqslant tq(x,t), \quad \text{for } (x,t) \in D \times (0,\infty),$$

where q is a nonnegative measurable function on $D \times (0,\infty)$ such that the function $t \to q(x,t)$ is nondecreasing on $(0,\infty)$ and $\lim_{t\to 0} q(x,t) = 0$.

(H_2) The function ψ defined on D by $\psi(x) = q(x, G(x,0))$ belongs to the class K.

Our main result is the following.

Theorem 1.1. *Assume* (H_1)–(H_2). *Then the problem* (P) *has infinitely many solutions. More precisely, there exists* $\lambda_0 > 0$ *such that for each* $\lambda \in (0,\lambda_0]$, *there exists a solution u of* (P) *continuous on* $D \setminus \{0\}$ *and satisfying*

$$\frac{\lambda}{2} G(x,0) \leqslant u(x) \leqslant \frac{3\lambda}{2} G(x,0),\ \forall x \in D \setminus \{0\} \quad \textit{and} \quad \lim_{|x|\to 0} \frac{u(x)}{G(x,0)} = \lambda.$$

Our paper is organized as follows. In Section 2, we introduce the Kato class K and we study the properties of functions belonging to this class. In Section 3, we prove Theorem 1.1.

2. The Kato class *K*

Throughout this paper, the letter C will denote a generic positive constant and D^c will denote the complementary of D in R^2.

Proposition 2.1. *Let* φ *be a function in K. Then the function* $y \to \rho^2(y)\varphi(y)$ *is in* $L^1(D)$.

Proof. Since $\varphi \in K$, then by (1.1) there exists $\alpha \in (0,1)$ such that

$$\sup_{x\in D} \int\limits_{(|x-y|\leqslant\alpha)\cap D} \frac{\rho(y)}{\rho(x)} \operatorname{Log}\left(1 + \frac{\rho(x)\rho(y)}{|x-y|^2}\right) \left|\varphi(y)\right| dy \leqslant 1.$$

Let $x_1, \ldots, x_n$ in D such that $D \subset \bigcup_{1\leqslant i\leqslant n} B(x_i,\alpha)$. Since

$$\operatorname{Log}(1+t) \geqslant \frac{t}{1+t}, \quad \forall t \geqslant 0,$$

then for all $i \in \{1,\ldots,n\}$ and $y \in B(x_i,\alpha) \cap D$, we have

$$\rho^2(y) \leqslant \left(\alpha^2 + d^2\right) \frac{\rho(y)}{\rho(x_i)} \operatorname{Log}\left(1 + \frac{\rho(x_i)\rho(y)}{|x_i - y|^2}\right),$$

where d is the diameter of D. Hence, we have

$$\begin{aligned}
&\int_D \rho^2(y)\big|\varphi(y)\big|\,dy\\
&\quad\leqslant \big(\alpha^2+d^2\big)\sum_{1\leqslant i\leqslant n}\ \int_{D\cap(|x_i-y|\leqslant\alpha)} \frac{\rho(y)}{\rho(x_i)}\mathrm{Log}\left(1+\frac{\rho(x_i)\rho(y)}{|x_i-y|^2}\right)\big|\varphi(y)\big|\,dy\\
&\quad\leqslant n\big(\alpha^2+d^2\big)\sup_{x\in D}\int_{D\cap(|x-y|\leqslant\alpha)} \frac{\rho(y)}{\rho(x)}\mathrm{Log}\left(1+\frac{\rho(x)\rho(y)}{|x-y|^2}\right)\big|\varphi(y)\big|\,dy\\
&\quad\leqslant n\big(\alpha^2+d^2\big)<\infty. \qquad\square
\end{aligned}$$

In the sequel, we use the notation

$$\|\varphi\|_D=\sup_{x\in D}\int_D \frac{\rho(y)}{\rho(x)}\mathrm{Log}\left(1+\frac{\rho(x)\rho(y)}{|x-y|^2}\right)\big|\varphi(y)\big|\,dy. \tag{2.1}$$

Proposition 2.2. *If $\varphi\in K$ then $\|\varphi\|_D<\infty$.*

Proof. Let $\alpha>0$. Then we have

$$\begin{aligned}
&\int_D \frac{\rho(y)}{\rho(x)}\mathrm{Log}\left(1+\frac{\rho(x)\rho(y)}{|x-y|^2}\right)\big|\varphi(y)\big|\,dy\\
&\quad\leqslant \int_{D\cap(|x-y|\leqslant\alpha)} \frac{\rho(y)}{\rho(x)}\mathrm{Log}\left(1+\frac{\rho(x)\rho(y)}{|x-y|^2}\right)\big|\varphi(y)\big|\,dy\\
&\qquad+\int_{D\cap(|x-y|\geqslant\alpha)} \frac{\rho(y)}{\rho(x)}\mathrm{Log}\left(1+\frac{\rho(x)\rho(y)}{|x-y|^2}\right)\big|\varphi(y)\big|\,dy.
\end{aligned}$$

Since

$$\int_{D\cap(|x-y|\geqslant\alpha)} \frac{\rho(y)}{\rho(x)}\mathrm{Log}\left(1+\frac{\rho(x)\rho(y)}{|x-y|^2}\right)\big|\varphi(y)\big|\,dy\leqslant\frac{1}{\alpha^2}\int_D \rho^2(y)\big|\varphi(y)\big|\,dy,$$

then the result follows immediatly from (1.1) and Proposition 2.1. $\square$

Let $G(x,y)$ be the Green's function for D corresponding to the Laplacian Δ. Then by [2] and [4], there exists $C>0$ such that for $x,y\in D$

$$\frac{1}{C}\mathrm{Log}\left(1+\frac{\rho(x)\rho(y)}{|x-y|^2}\right)\leqslant G(x,y)\leqslant C\,\mathrm{Log}\left(1+\frac{\rho(x)\rho(y)}{|x-y|^2}\right) \tag{2.2}$$

and

$$\frac{\rho(y)}{\rho(x)}G(x,y)\leqslant C\big(1+G(x,y)\big). \tag{2.3}$$

Furthermore, G_D satisfies the 3G-Theorem [4]:

3G-Theorem. *There exists a constant C_0 depending only on D such that for all x, y and z in D, we have*

$$\frac{G(x,z)G(z,y)}{G(x,y)} \leqslant C_0\left[\frac{\rho(z)}{\rho(x)}G(x,z) + \frac{\rho(z)}{\rho(y)}G(z,y)\right]. \tag{2.4}$$

By using the above theorem, we have

Proposition 2.3. *There exists a constant $C_D > 0$ depending only on D such that for any function φ belonging to K, any nonnegative superharmonic function h in D and all $x \in D$*

$$\int_D G(x,y)h(y)\left|\varphi(y)\right| dy \leqslant C_D \|\varphi\|_D h(x). \tag{2.5}$$

Proof. Let h be a nonnegative superharmonic function in D. Then there exists a sequence $(f_n)_n$ of nonnegative measurable functions in D such that

$$h(y) = \sup_n \int_D G(y,z) f_n(z)\, dz.$$

Hence, we need only to verify (2.5) for $h(y) = G(y,z)$, $\forall z \in D$. By using (2.4) and (2.2), there exists $C_D > 0$ such that

$$\begin{aligned}
&\frac{1}{G(x,z)} \int_D G(x,y)G(y,z)\left|\varphi(y)\right| dy \\
&\quad \leqslant C_0 \int_D \left[\frac{\rho(y)}{\rho(x)}G(x,y) + \frac{\rho(y)}{\rho(z)}G(z,y)\right]\left|\varphi(y)\right| dy \\
&\quad \leqslant C_D \|\varphi\|_D. \qquad \square
\end{aligned}$$

Corollary 2.4. *Let φ be a function in K. Then*

$$\sup_{x\in D} \int_D G(x,y)\left|\varphi(y)\right| dy < \infty. \tag{2.6}$$

Corollary 2.5. *Let φ be a function in K. Then the function $y \to \rho(y)\varphi(y)$ is in $L^1(D)$.*

Proof. Since D is bounded and $\mathrm{Log}(1+t) \geqslant \frac{t}{1+t}$, $\forall t \geqslant 0$, then by (2.2), there exists $C > 0$ such that

$$\rho(x)\rho(y) \leqslant C G(x,y), \quad \forall x, y \in D.$$

Let $x_0 \in D$. Then we have

$$\int_D \rho(y)\big|\varphi(y)\big|\,dy \leqslant \frac{C}{\rho(x_0)} \int_D G(x_0, y)\big|\varphi(y)\big|\,dy.$$

Thus, the result follows from Corollary 2.4. □

In the next proposition, we prove in the case where φ is radial and D is a ball or a annulus that φ is in the class K if and only if (2.6) is satisfied.

Proposition 2.6. (i) *Let φ be a radial function in $B(0,1)$. Then the function φ is in the class K if and only if*

$$\int_0^1 r\,\mathrm{Log}\left(\frac{1}{r}\right)\big|\varphi(r)\big|\,dr < \infty. \tag{2.7}$$

(ii) *Let φ be a radial function in $\{x \in R^2\colon\ a < |x| < b\}$, for some $b > a > 0$. Then the function φ is in the class K if and only if*

$$\int_a^b (b-r)(r-a)\big|\varphi(r)\big|\,dr < \infty. \tag{2.8}$$

Proof. (i) The Green's function of $B(0,1)$ is given by

$$G(x,y) = \frac{1}{2\pi}\mathrm{Log}\left(1 + \frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}\right).$$

By elementary calculus, we have

$$\int_D G(x,y)\big|\varphi(y)\big|\,dy = \int_0^1 r\,\mathrm{Log}\left(\frac{1}{r \vee t}\right)\big|\varphi(r)\big|\,dr,$$

where $t = |x|$ and $t \vee r = \max(t,r)$. Hence by (2.6) we deduce that if φ is in the class K then

$$\int_0^1 r\,\mathrm{Log}\left(\frac{1}{r}\right)\big|\varphi(r)\big|\,dr < \infty.$$

To prove the converse, suppose that φ satisfies $\int_0^1 r\,\mathrm{Log}(\frac{1}{r})|\varphi(r)|\,dr < \infty$. Then by (2.2) there exists $C > 0$ such that for $\alpha > 0$,

$$\int_{D\cap(|x-y|\leqslant\alpha)} \frac{1-|y|}{1-|x|}\mathrm{Log}\left(1 + \frac{\rho(x)\rho(y)}{|x-y|^2}\right)\big|\varphi(y)\big|\,dy$$

$$\leqslant C \int_{(0,1)\cap(|t-r|\leqslant\alpha)} \frac{1-r}{1-t} r \operatorname{Log}\left(\frac{1}{r \vee t}\right) \left|\varphi(r)\right| dr.$$

Let $0 < \alpha < \frac{1}{4}$. Since $1 - r \leqslant \operatorname{Log}\frac{1}{r}$ on $[\frac{1}{4}, 1]$, then we have

$$\sup_{0\leqslant t\leqslant 1} \int_{]t-\alpha,t+\alpha[\cap]0,1[} \frac{1-r}{1-t} r \operatorname{Log}\left(\frac{1}{r\vee t}\right)\left|\varphi(r)\right| dr$$

$$\leqslant 2 \sup_{0\leqslant t\leqslant \frac{1}{2}} \int_{(t-\alpha)\vee 0}^{(t+\alpha)\wedge 1} r \operatorname{Log}\left(\frac{1}{r}\right)\left|\varphi(r)\right| dr$$

$$+ \sup_{\frac{1}{2}\leqslant t\leqslant 1} \frac{\operatorname{Log}(\frac{1}{t})}{1-t} \int_{t-\alpha}^{(t+\alpha)\wedge 1} r \operatorname{Log}\left(\frac{1}{r}\right)\left|\varphi(r)\right| dr$$

$$\leqslant 4 \sup_{0\leqslant t\leqslant 1} \int_{(t-\alpha)\vee 0}^{(t+\alpha)\wedge 1} r \operatorname{Log}\left(\frac{1}{r}\right)\left|\varphi(r)\right| dr,$$

where $(t+\alpha)\wedge 1 = \min(t+\alpha, 1)$.

Let $\phi(x) = \int_0^x r \operatorname{Log}(\frac{1}{r})|\varphi(r)|\,dr$, $x \in [0,1]$. By (2.7), ϕ is a continuous function on $[0,1]$. Hence

$$\int_{(t-\alpha)\vee 0}^{(t+\alpha)\wedge 1} r \operatorname{Log}\left(\frac{1}{r}\right)\left|\varphi(r)\right| dr = \phi\big((t+\alpha)\wedge 1\big) - \phi\big((t-\alpha)\vee 0\big)$$

converges to zero as $\alpha \to 0$ uniformly for $t \in [0,1]$.

(ii) Let G be the Green's function of $D = \{x \in R^2\colon\ a < |x| < b\}$. Then we have

$$\int_{a<|y|<b} G(x,y)\left|\varphi(y)\right| dy$$

$$= \frac{1}{\operatorname{Log}\frac{b}{a}} \int_a^b r \operatorname{Log}\left(\frac{r\wedge t}{a}\right) \operatorname{Log}\left(\frac{b}{r\vee t}\right)\left|\varphi(r)\right| dr, \tag{2.9}$$

where $t = |x|$. Moreover, we have

$$\frac{a}{2b^2}\int_a^b (b-r)(r-a)\left|\varphi(r)\right| dr$$

$$\leqslant \sup_{a\leqslant t\leqslant b} \int_a^b r \operatorname{Log}\left(\frac{r\wedge t}{a}\right) \operatorname{Log}\left(\frac{b}{r\vee t}\right)\left|\varphi(r)\right| dr$$

$$\leqslant \frac{b}{a^2}\int_a^b (b-r)(r-a)\bigl|\varphi(r)\bigr|\,dr.$$

Conversely, let $\alpha > 0$. Since

$$\frac{(|y|-a)(b-|y|)}{b-a} \leqslant \rho(y) \leqslant 2\frac{(|y|-a)(b-|y|)}{b-a},$$

then by (2.2) there exists $C > 0$ such that

$$\begin{aligned}
&\int_{D\cap(|x-y|\leqslant\alpha)} \frac{\rho(y)}{\rho(x)}\mathrm{Log}\biggl(1+\frac{\rho(x)\rho(y)}{|x-y|^2}\biggr)\bigl|\varphi(y)\bigr|\,dy \\
&\quad\leqslant C\int_{(|t-r|\leqslant\alpha)\cap(a,b)} \frac{(r-a)(b-r)}{(t-a)(b-t)} r\,\mathrm{Log}\biggl(\frac{r\wedge t}{a}\biggr)\mathrm{Log}\biggl(\frac{b}{r\vee t}\biggr)\bigl|\varphi(r)\bigr|\,dr \\
&\quad\leqslant Cb\frac{\mathrm{Log}(\frac{t}{a})\mathrm{Log}(\frac{b}{t})}{(t-a)(b-t)}\int_{(t-\alpha)\vee a}^{(t+\alpha)\wedge b} (r-a)(b-r)\bigl|\varphi(r)\bigr|\,dr \\
&\quad\leqslant C\frac{b}{a^2}\int_{(t-\alpha)\vee a}^{(t+\alpha)\wedge b} (r-a)(b-r)\bigl|\varphi(r)\bigr|\,dr
\end{aligned}$$

which converges to 0 when $\alpha \to 0$ uniformly for $t \in [a, b]$. $\square$

Remark 2.1. Let φ be the function defined on $B(0, 1)$ by

$$\varphi(y) = \frac{1}{(1-|y|)^2}.$$

Then it is clear that $\varphi \in L^1(B(0,1), (1-|y|)^2\,dy)$. On the other hand, by (2.7) we deduce that $\varphi \notin K$. Hence, $L^1(B(0,1), \rho^2(y)\,dy)$ properly contains the class K.

Proposition 2.7. *The class K properly contains $L^p(D)$ class with $1 < p \leqslant \infty$.*

Proof. Let $\varphi \in L^p(D)$, for $1 < p \leqslant \infty$. Let $\alpha > 0$ and $d = \mathrm{diam}(D)$. By (2.2) and (2.3) there exists $C > 0$ such that

$$\begin{aligned}
&\int_{D\cap(|x-y|\leqslant\alpha)} \frac{\rho(y)}{\rho(x)}\mathrm{Log}\biggl(1+\frac{\rho(x)\rho(y)}{|x-y|^2}\biggr)\bigl|\varphi(y)\bigr|\,dy \\
&\quad\leqslant C\int_{D\cap(|x-y|\leqslant\alpha)} \biggl(1+\mathrm{Log}\biggl(1+\frac{\rho(x)\rho(y)}{|x-y|^2}\biggr)\biggr)\bigl|\varphi(y)\bigr|\,dy
\end{aligned}$$

$$\leqslant C\|\varphi\|_p \left[\int_0^{\alpha^2} \left(1 + \mathrm{Log}\left(1 + \frac{d^2}{t}\right)\right)^{\frac{p}{p-1}} dt \right]^{\frac{p-1}{p}},$$

which converges to zero when $\alpha \to 0$. $\square$

Remark 2.2. Let φ be the function defined on $B(0, \frac{1}{2})$ by

$$\varphi(y) = \frac{1}{|y|^2 |\mathrm{Log}|y||^{3/2}}.$$

Then $\varphi \in L^1(B(0, \frac{1}{2}))$, but from (2.7) $\varphi \notin K$.

Proposition 2.8. *Let $\lambda < 2$. Then the function defined in D by*

$$\rho_\lambda(y) = \frac{1}{(\rho(y))^\lambda}$$

is in the class K.

To prove this proposition, we need the next lemma (see [3]).

Lemma 2.9. *Let $x, y \in D$. Then we have the following properties*:

(i) *If $\rho(x)\rho(y) \leqslant |x-y|^2$ then*

$$\max\bigl(\rho(x), \rho(y)\bigr) \leqslant \frac{\sqrt{5}+1}{2}|x-y|.$$

(ii) *If $|x-y|^2 \leqslant \rho(x)\rho(y)$ then for each $z \in D^c$,*

$$\frac{3-\sqrt{5}}{2}|x-z| \leqslant |y-z| \leqslant \frac{3+\sqrt{5}}{2}|x-z|.$$

In particular, we have

$$\frac{3-\sqrt{5}}{2}\rho(x) \leqslant \rho(y) \leqslant \frac{3+\sqrt{5}}{2}\rho(x).$$

Proof. (i) We may assume that $\max(\rho(x), \rho(y)) = \rho(y)$. Then the inequalities $\rho(y) \leqslant \rho(x) + |x-y|$ and $\rho(x)\rho(y) \leqslant |x-y|^2$ imply that

$$\bigl(\rho(y)\bigr)^2 - \rho(y)|x-y| - |x-y|^2 \leqslant 0,$$

i.e.,

$$\left[\rho(y) + \frac{\sqrt{5}-1}{2}|x-y|\right]\left[\rho(y) - \frac{\sqrt{5}+1}{2}|x-y|\right] \leqslant 0.$$

It follows that

$$\max\bigl(\rho(x),\rho(y)\bigr) \leqslant \frac{\sqrt{5}+1}{2}|x-y|.$$

(ii) For each $z \in D^c$, we have $|y-z| \leqslant |x-y| + |x-z|$ and since $|x-y|^2 \leqslant \rho(x)\rho(y)$, we obtain

$$|y-z| \leqslant \sqrt{\rho(x)\rho(y)} + |x-z| \leqslant \sqrt{|x-z||y-z|} + |x-z|,$$

i.e.,

$$\left[\sqrt{|y-z|} + \frac{\sqrt{5}-1}{2}\sqrt{|x-z|}\right]\left[\sqrt{|y-z|} - \frac{\sqrt{5}+1}{2}\sqrt{|x-z|}\right] \leqslant 0.$$

It follows that

$$|y-z| \leqslant \frac{\sqrt{5}+3}{2}|x-z|.$$

Thus, interchange the role of x and y, we have

$$|x-z| \leqslant \frac{\sqrt{5}+3}{2}|y-z|.$$

Which implies that

$$\frac{3-\sqrt{5}}{2}|x-z| \leqslant |y-z| \leqslant \frac{3+\sqrt{5}}{2}|x-z|.$$

In particular,

$$\frac{3-\sqrt{5}}{2}\inf_{z\in\partial D}|x-z| \leqslant \inf_{z\in\partial D}|y-z| \leqslant \frac{3+\sqrt{5}}{2}\inf_{z\in\partial D}|x-z|.$$

Which gives

$$\frac{3-\sqrt{5}}{2}\rho(x) \leqslant \rho(y) \leqslant \frac{3+\sqrt{5}}{2}\rho(x). \qquad \square$$

Proof of Proposition 2.8. If $\lambda \leqslant 0$, then $\rho_\lambda \in L^\infty(D)$. Hence by Proposition 2.7, $\rho_\lambda \in K$.

Let $0 < \lambda < 2$. Then

$$\begin{aligned}
&\int_{D\cap(|x-y|\leqslant\alpha)} \frac{\rho(y)}{\rho(x)}\mathrm{Log}\left(1+\frac{\rho(x)\rho(y)}{|x-y|^2}\right)\frac{1}{(\rho(y))^\lambda}\,dy \\
&\quad\leqslant \int_{D\cap(|x-y|\leqslant\alpha)\cap(\rho(x)\rho(y)\leqslant|x-y|^2)} \frac{\rho(y)}{\rho(x)}\mathrm{Log}\left(1+\frac{\rho(x)\rho(y)}{|x-y|^2}\right)\frac{1}{(\rho(y))^\lambda}\,dy \\
&\quad\quad+ \int_{D\cap(|x-y|\leqslant\alpha)\cap(|x-y|^2\leqslant\rho(x)\rho(y))} \frac{\rho(y)}{\rho(x)}\mathrm{Log}\left(1+\frac{\rho(x)\rho(y)}{|x-y|^2}\right)\frac{1}{(\rho(y))^\lambda}\,dy \\
&\quad= I_1 + I_2.
\end{aligned}$$

So, using Lemma 2.9, we obtain

$$I_1 \leqslant \int\limits_{D\cap(|x-y|\leqslant\alpha)\cap(\rho(x)\vee\rho(y)\leqslant\frac{\sqrt{5}+1}{2}|x-y|)} \frac{(\rho(y))^{2-\lambda}}{|x-y|^2}\,dy$$
$$\leqslant C\int_0^{\alpha} t^{1-\lambda}\,dt \leqslant C\alpha^{2-\lambda}$$

which converges to zero as $\alpha \to 0$.

On the other hand, put $\delta(x) = \frac{\sqrt{5}+1}{2}\rho(x)$, then using Lemma 2.9, we have

$$I_2 \leqslant \frac{C}{(\rho(x))^{\lambda}} \int\limits_{D\cap(|x-y|\leqslant\alpha)\cap(|x-y|\leqslant\delta(x))} \operatorname{Log}\left(1+\frac{(\delta(x))^2}{|x-y|^2}\right)dy$$
$$\leqslant C\big(\rho(x)\big)^{2-\lambda} \int_0^{\frac{\alpha^2}{(\delta(x))^2}\wedge 1} \operatorname{Log}\left(1+\frac{1}{r}\right)dr.$$

Which implies that if $\delta(x) \leqslant \alpha$ then

$$I_2 \leqslant C\big(\delta(x)\big)^{2-\lambda} \leqslant C\alpha^{2-\lambda} \to 0 \quad \text{as } \alpha \to 0,$$

and if $\alpha \leqslant \delta(x)$ we have

$$I_2 \leqslant C\frac{\alpha^{2-\lambda}}{t^{1-\frac{\lambda}{2}}}\int_0^t \operatorname{Log}\left(1+\frac{1}{r}\right)dr$$
$$\leqslant C\alpha^{2-\lambda} t^{\frac{\lambda}{2}}\left(\operatorname{Log}(1+t) - \operatorname{Log} t + \frac{\operatorname{Log}(1+t)}{t}\right),$$

where $t = \alpha^2/(\delta(x))^2$. Since $\lambda > 0$, then

$$\lim_{t\to 0} t^{\frac{\lambda}{2}}\left(\operatorname{Log}(1+t) - \operatorname{Log} t + \frac{\operatorname{Log}(1+t)}{t}\right) = 0.$$

Thus

$$I_2 \leqslant C\alpha^{2-\lambda} \to 0 \quad \text{as } \alpha \to 0. \qquad \square$$

Now we shall compare the class K and the Kato class K_2.

Definition 2.1 (*see* [1] *or* [2]). A Borel measurable function φ in D belongs to the Kato class K_2 if

$$\lim_{\alpha\to 0}\left(\sup_{x\in R^2} \int\limits_{(|x-y|\leqslant\alpha)\cap D} \operatorname{Log}\left(\frac{1}{|x-y|}\right)\big|\varphi(y)\big|\,dy\right) = 0. \tag{2.10}$$

Remark 2.3. Let $D = B(0,1)$ and $1 < \lambda < 2$. Then $\rho_\lambda \notin K_2$. In fact, $K_2 \subset L^1(D)$ and for $\lambda > 1$,

$$\int_D \rho_\lambda(y)\,dy = 2\pi \int_0^1 \frac{r}{(1-r)^\lambda}\,dr = \infty.$$

Proposition 2.10. *The class K properly contains the classical Kato class K_2.*

Proof. Let $\varphi \in K_2$. Then for all $\varepsilon > 0$, there exists $\alpha_1 > 0$ such that

$$\sup_{x\in D} \int_{(|x-y|\leqslant\alpha_1)\cap D} \mathrm{Log}\left(\frac{1}{|x-y|}\right)\bigl|\varphi(y)\bigr|\,dy \leqslant \varepsilon.$$

Since D is bounded, then by (2.2) and (2.3), there exists $C > 0$ such that for $x, y \in D$

$$\frac{\rho(y)}{\rho(x)}\mathrm{Log}\left(1 + \frac{\rho(x)\rho(y)}{|x-y|^2}\right) \leqslant C\left(1 + \mathrm{Log}\frac{1}{|x-y|}\right).$$

Let $\alpha \leqslant \min(\alpha_1, \frac{1}{e})$. Then for $x \in D$,

$$\int_{(|x-y|\leqslant\alpha)\cap D} \frac{\rho(y)}{\rho(x)}\mathrm{Log}\left(1 + \frac{\rho(x)\rho(y)}{|x-y|^2}\right)\bigl|\varphi(y)\bigr|\,dy$$
$$\leqslant C \sup_{x\in D} \int_{(|x-y|\leqslant\alpha)\cap D} \mathrm{Log}\left(\frac{1}{|x-y|}\right)\bigl|\varphi(y)\bigr|\,dy \leqslant C\varepsilon. \qquad \square$$

Remark 2.4. (1) Let $p \in\,]1, \infty]$. Then we have

$$L^p(D) \subset K_2 \subset K \cap L^1(D) \subset K \subset L^1_{\mathrm{loc}}(D).$$

(2) In case that φ is radial and D is a ball, we prove that

$$K_2 = K \cap L^1(D).$$

3. Proof of Theorem 1.1

Before proving the theorem, we need the following lemmas:

Lemma 3.1. *Let $\alpha > 0$. Then there exists $C > 0$ such that for all $x, y \in D$ satisfying $|x-y| \geqslant \alpha$, we have*

$$\frac{\rho(y)}{\rho(x)}G(x,y) \leqslant C\rho^2(y). \tag{3.1}$$

Moreover, if $|y| \geqslant \alpha$ *then*

$$\frac{G(x,y)G(y,0)}{G(x,0)} \leqslant C\rho^2(y). \tag{3.2}$$

Proof. This lemma follows immediately from (2.2) and the 3G-Theorem. □

Lemma 3.2. *Let* $x_0 \in \overline{D}$. *Then for any function* φ *belonging to K and any positive superharmonic function h in D, we have*

$$\lim_{\delta\to 0}\left(\sup_{x\in D}\frac{1}{h(x)}\int_{B(x_0,\delta)\cap D} G(x,y)h(y)\left|\varphi(y)\right|dy\right) = 0. \tag{3.3}$$

Proof. Let h be a positive superharmonic function in D. Then there exists a sequence $(f_n)_n$ of nonnegative measurable functions in D such that

$$h(y) = \sup_n \int_D G(y,z) f_n(z)\,dz.$$

Hence, it is enough to verify (3.3) for $h(y) = G(y,z)$, uniformly for $z \in D$.

Let $\delta > 0$. By using the 3G-Theorem, we obtain

$$\begin{aligned}&\frac{1}{G(x,z)}\int_{B(x_0,\delta)\cap D} G(x,y)G(y,z)\left|\varphi(y)\right|dy\\ &\quad\leqslant 2C_0 \sup_{x\in D}\int_{B(x_0,\delta)\cap D}\frac{\rho(y)}{\rho(x)}G(x,y)\left|\varphi(y)\right|dy.\end{aligned}$$

Let $\alpha > 0$. Then using (3.1), we have

$$\begin{aligned}&\int_{B(x_0,\delta)\cap D}\frac{\rho(y)}{\rho(x)}G(x,y)\left|\varphi(y)\right|dy\\ &\quad\leqslant \int_{D\cap(|x-y|\leqslant\alpha)}\frac{\rho(y)}{\rho(x)}G(x,y)\left|\varphi(y)\right|dy\\ &\qquad+\int_{B(x_0,\delta)\cap D\cap(|x-y|\geqslant\alpha)}\frac{\rho(y)}{\rho(x)}G(x,y)\left|\varphi(y)\right|dy\\ &\quad\leqslant \int_{D\cap(|x-y|\leqslant\alpha)}\frac{\rho(y)}{\rho(x)}G(x,y)\left|\varphi(y)\right|dy + C\int_{B(x_0,\delta)\cap D}\rho^2(y)\left|\varphi(y)\right|dy.\end{aligned}$$

Since φ is in K, it follows from (2.2) and (1.1) that $\forall\varepsilon > 0$, we can determine some $\alpha > 0$ such that

$$\int\limits_{D\cap(|x-y|\leqslant\alpha)} \frac{\rho(y)}{\rho(x)} G(x,y)\big|\varphi(y)\big|\,dy \leqslant \varepsilon, \quad \forall x \in D.$$

Fixing this α, letting $\delta \to 0$, we obtain (3.3) by Proposition 2.1. $\square$

Let $C(\overline{D})$ be the space of all continuous functions in $\overline{D}$ endowed with the uniform norm $\|\cdot\|_\infty$. Let

$$F := \big\{w \in C^+(\overline{D}):\ \|w\|_\infty \leqslant 1\big\}.$$

Lemma 3.3. *Assume* (H$_1$)–(H$_2$). *Let T be the operator defined on F by*

$$Tw(x) = \frac{1}{G(x,0)} \int\limits_D G(x,y) f\big(y, w(y)G(y,0)\big)\,dy, \quad x \in D.$$

Then the family of functions $T(F) = \{T(w),\ w \in F\}$ is relatively compact in $C(\overline{D})$.

Proof. By (H$_1$), we have for all $w \in F$

$$\big|Tw(x)\big| \leqslant \frac{1}{G(x,0)} \int\limits_D G(x,y)G(y,0)\psi(y)\,dy.$$

Since $\psi \in K$, then from Proposition 2.3, we deduce that

$$\|Tw\|_\infty \leqslant C_D\|\psi\|_D.$$

Thus the family $T(F)$ is uniformly bounded. Now, we propose to prove the equicontinuity of $T(F)$ in $C(\overline{D})$.

Let $x_0 \in \overline{D}$ and $\delta > 0$. Let $x, x' \in B(x_0, \frac{\delta}{2}) \cap D$ and $w \in F$, then

$$\begin{aligned}
&\big|Tw(x) - Tw(x')\big| \\
&\quad\leqslant 2 \sup_{x\in D} \frac{1}{G(x,0)} \int\limits_{B(0,\delta)\cap D} G(x,y)G(y,0)\psi(y)\,dy \\
&\qquad + 2 \sup_{x\in D} \frac{1}{G(x,0)} \int\limits_{B^c(0,\delta)\cap B(x_0,\delta)\cap D} G(x,y)G(y,0)\psi(y)\,dy \\
&\qquad + \int\limits_{B^c(0,\delta)\cap B^c(x_0,\delta)\cap D} \left|\frac{G(x,y)}{G(x,0)} - \frac{G(x',y)}{G(x',0)}\right| G(y,0)\psi(y)\,dy.
\end{aligned}$$

By Lemma 3.1, there exists $C > 0$ such that for all $x \in B(x_0, \frac{\delta}{2}) \cap D$, and $y \in B^c(0,\delta) \cap D \setminus B(x_0,\delta) = D_0$

$$\frac{G(x,y)}{G(x,0)} G(y,0)\psi(y) \leqslant C\rho^2(y)\psi(y).$$

Moreover, $G(x,y)/G(x,0)$ is continuous on $(x,y)\in(B(x_0,\frac{\delta}{2})\cap\overline{D})\times D_0$, and $G(y,0)$ has no singularities when $y\in B^c(0,\delta)\cap D$. Then by Proposition 2.1 and Lebesgue's Theorem, we have

$$\int\limits_{B^c(0,\delta)\cap B^c(x_0,\delta)\cap D}\left|\frac{G(x,y)}{G(x,0)}-\frac{G(x',y)}{G(x',0)}\right|G(y,0)\psi(y)\,dy\to 0$$

as $|x-x'|\to 0$. Then it follows from Lemma 3.2 that

$$\left|Tw(x)-Tw(x')\right|\to 0\quad\text{as } |x-x'|\to 0$$

uniformly for all $w\in F$.

Finally, by Ascoli's Theorem, the family $T(F)$ is relatively compact in $C(\overline{D})$. $\square$

Proof of Theorem 1.1. Let $\beta\in(0,1)$. Then the function

$$T_\beta(x)=\frac{1}{G(x,0)}\int\limits_D G(x,y)G(y,0)q\bigl(y,\beta G(y,0)\bigr)\,dy$$

is continuous in $\overline{D}$ satisfying

$$\forall x\in\overline{D},\ \lim_{\beta\to 0}T_\beta(x)=0.$$

Moreover, the function $\beta\to T_\beta(x)$ is nondecreasing in $(0,1)$. Then, by Dini Lemma, we have

$$\lim_{\beta\to 0}\left(\sup_{x\in\overline{D}}\frac{1}{G(x,0)}\int\limits_D G(x,y)G(y,0)q\bigl(y,\beta G(y,0)\bigr)\,dy\right)=0.$$

Thus, there exists $\beta\in(0,1)$ such that for each $x\in\overline{D}$,

$$\frac{1}{G(x,0)}\int\limits_D G(x,y)G(y,0)q\bigl(y,\beta G(y,0)\bigr)\,dy\leqslant\frac{1}{3}.$$

Let $\lambda_0=\frac{2}{3}\beta$ and $\lambda\in(0,\lambda_0]$. In order to apply a fixing point argument, set

$$S=\left\{w\in C(\overline{D}):\ \frac{\lambda}{2}\leqslant w(x)\leqslant\frac{3\lambda}{2},\ x\in D\right\}.$$

Then, S is a nonempty closed bounded and convex set in $C(\overline{D})$.

Define the operator Γ on S by

$$\Gamma w(x)=\lambda+\frac{1}{G(x,0)}\int\limits_D G(x,y)f\bigl(y,w(y)G(y,0)\bigr)\,dy,\quad x\in D.$$

First, we shall prove that the operator Γ maps S into itself. Let $w \in S$, then for any $x \in D$, we have

$$\left|\Gamma w(x) - \lambda\right| \leqslant \frac{3\lambda}{2} \frac{1}{G(x,0)} \int_D G(x,y)G(y,0)q\big(y, \beta G(y,0)\big)\,dy \leqslant \frac{\lambda}{2}.$$

It follows that $\frac{\lambda}{2} \leqslant \Gamma w \leqslant \frac{3\lambda}{2}$ and by Lemma 3.3, $\Gamma(S)$ is included in $C(\overline{D})$. So $\Gamma(S) \subset S$.

Next, we shall prove the continuity of Γ in the supremum norm. Let $(w_k)_k$ be a sequence in S which converges uniformly to $w \in S$. It follows from (H_1) and Lebesgue's Theorem that

$$\forall x \in D,\ \Gamma w_k(x) \to \Gamma w(x) \quad \text{as } k \to \infty.$$

Since $\Gamma(S)$ is a relatively compact family in $C(\overline{D})$, then the pointwise convergence implies the uniform convergence. Thus we have proved that Γ is a compact mapping from S to itself. Now the Schauder fixed point theorem implies the existence of $w \in S$ such that $\Gamma w = w$.

For all $x \in D$, put $u(x) = w(x)G(x,0)$. Therefore

$$u(x) = \lambda G(x,0) + \int_D G(x,y) f\big(y, u(y)\big)\,dy.$$

Since $G(x,0) \sim \operatorname{Log}\frac{1}{|x|}$ near $x = 0$ then it is clear that u is a solution of (P), continuous on $D \setminus \{0\}$ satisfying

$$\frac{\lambda}{2} G(x,0) \leqslant u(x) \leqslant \frac{3\lambda}{2} G(x,0) \quad \text{and} \quad \lim_{|x| \to 0} \frac{u(x)}{G(x,0)} = \lambda. \qquad \square$$

Example 3.1. Let $p > 1$ and V be a measurable function in $B(0,1)$. Suppose that there exists a nonnegative measurable function k on $(0,1)$ such that

$$\left|V(x)\right| \leqslant k\big(|x|\big), \quad \forall x \in B(0,1),$$

and

$$\int_0^1 r \operatorname{Log}\left(\frac{1}{r}\right)^p k(r)\,dr < \infty.$$

Then there exists $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0]$, the problem

$$(\mathrm{P}) \begin{cases} \Delta u(x) + V(x) u^p(x) = 0,\ x \in B(0,1) \setminus \{0\}, \\ u(x) > 0,\ x \in B(0,1) \setminus \{0\}, \\ u(x) = 0,\ x \in \partial B \end{cases}$$

has a solution u continuous on $B(0,1)\setminus\{0\}$ such that

$$\lim_{|x| \to 0} \frac{u(x)}{\operatorname{Log}\frac{1}{|x|}} = \lambda.$$

Example 3.2. Let $\gamma \in R$, $\alpha > \max(0, \gamma)$ and $\beta > 0$. Then there exists $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0]$, the problem

$$(\mathrm{P}) \begin{cases} \Delta u(x) + \dfrac{u^{\alpha+1}(x)}{|x|^\beta + u^\gamma(x)} = 0, \ x \in B(0,1) \setminus \{0\}, \\ u(x) > 0, \ x \in B(0,1) \setminus \{0\}, \\ u(x) = 0, \ x \in \partial B \end{cases}$$

has a solution u continuous on $B(0,1) \backslash \{0\}$ such that

$$\lim_{|x| \to 0} \frac{u(x)}{\operatorname{Log} \frac{1}{|x|}} = \lambda.$$

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